Developable surface patches bounded by NURBS curves

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Abstract

In this talk we revisit the problem of constructing a developable surface patch bounded by two rational or NURBS (Non-Uniform Rational B-spline) curves [1].

NURBS curves are curves which are piecewise rational. That is, they are a generalisation of spline curves, which are piecewise polynomial curves. Similarly we define NURBS surfaces and solids. NURBS curves are the standard in Computer Aided Design.

Developable surfaces are ruled surfaces with null Gaussian curvature. This implies that they can be constructed from planar surfaces by just cutting, rolling and folding, so that metric properties such as lengths and angles between curves and areas are preserved. These geometric properties are of great interest for steel and textile industry, since these are pieces designed in the plane and then curved into space.

For instance, in naval architecture sheets of steel are adapted to fit into the hull of a ship. In textile industry cloth is planar and is cut and sewn to produce garments and the quality is improved if it is not stretched.

This problem has been addressed in several ways, but the key drawback is that when we require the developable surface to be NURBS and bounded by NURBS curves, the possibilities are restricted [2].

For this reason our proposal is to consider developable surface patches which are not NURBS, though bounded by NURBS curves [1]. In fact, we are able to obtain every possible solution to this problem in our framework.

We start with a ruled surface parametrised by $b(t,v)$ and bounded by two parametrised curves, $c(t)$, $d(t)$,

$$b(t,v) = (1-v)c(t) + vd(t), \quad t, v \in [0,1].$$

For given $c(t)$ and $d(t)$, this ruled surface will not be developable in general. Our contribution to deal with this problem is based on reparametrisation of one of the bounding curves by a function $T(t)$,

$$\tilde{b}(t,v) = (1-v)c(t) + vd(T(t))$$

and require $\tilde{b}(t,v)$ to satisfy the null Gaussian curvature condition. This condition can be seen to be algebraic in $T(t)$, since the dependence on the derivative $T'(t)$ is factored out,

$$\det\left(c'(t), \dot{d}(T), d(T) - c(t)\right) = 0,$$

and is of degree $2n-2$ if the bounding curves $c(t)$, $d(t)$ are of degree n. The dot and the comma stand for derivation with respect to T and t.

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The price to pay is that solutions of this algebraic equation will not be rational or polynomial in general and $\tilde{b}(t,v)$ will no longer be NURBS.

There is an important case which is even simpler to handle. If the bounding curves $c(t)$, $d(t)$ are not rational or piecewise rational (just polynomial or piecewise polynomial) and lie on parallel planes, the degree may be seen to decrease to $n-1$.

Since the condition on the reparametrisation is algebraic, the number of possible solutions is finite, but not all of them are geometrically acceptable. For being a reparametrisation, $T(t)$ must be a monotonically increasing function. This can be checked with the help of

$$T'(t) = \left.\frac{\det\left(c''(t), \dot{d}(T), d(T) - c(t)\right)}{\det\left(\ddot{d}(T), c'(t), d(T) - c(t)\right)}\right|_{T=T(t)},$$

which we derive from the null Gaussian curvature condition.

This implies that monotonicity is granted if

$$\operatorname{sgn}\left(c''(t)\cdot\nu(t)\right) = \operatorname{sgn}\left(\ddot{d}(T)\cdot\nu(t)\right)\Big|_{T=T(t)},$$

where $\nu(t)$ is the unitary normal to the ruled surface along the segment at t. This means that the normal curvatures of both curves must have the same sign for each value of t.

Hence, acceptable solutions just appear if both curves are qualitatively similar regarding their curvature.

References

[1] Fernández-Jambrina, L., Pérez-Arribas, F., Developable surfaces bounded by NURBS curves, *Journal of Computational Mathematics* 38: 693–709, 2020.

[2] Fernández-Jambrina, L., Bézier developable surfaces, *Computer Aided Geometric Design* 55: 15–28, 2017.